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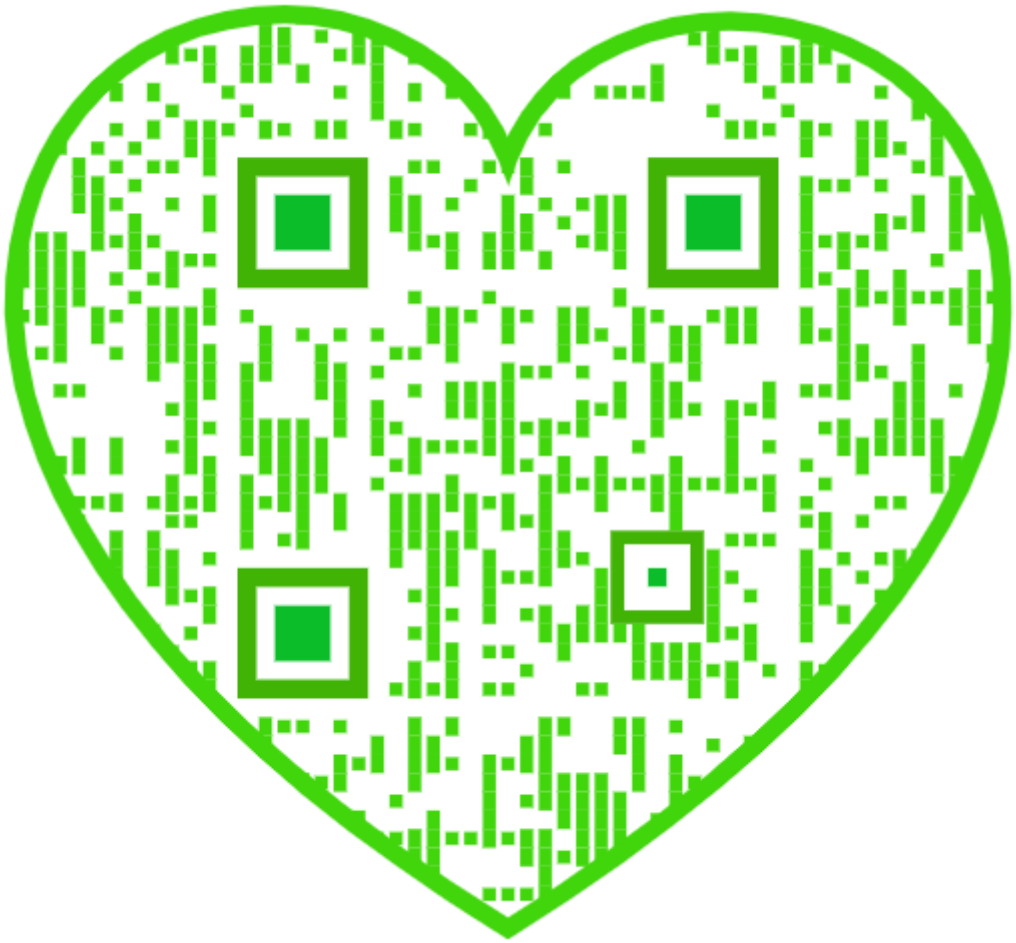
Grube's Method.



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How to Teach Elementary Arithmetic

GRUBE'S METHOD

OF

TEACHING ARITHMETIC EXPLAINED

WITH A LARGE NUMBER OF

PRACTICAL HINTS AND ILLUSTRATIONS

BY

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A. M. C.

PREFACE.

THE first of the following two essays is the same in substance as the one read before the St. Louis Teachers' Association in 1870, which has been republished since extensively in state and city school reports and educational magazines. It is presented here in a somewhat changed form, because the practical experience in the schoolroom has shown since what points of the method are in such harmony with established views as to require no further explanation, and what details need full comment and amplification in order to guard against such mistakes as are likely to creep in. In some respects I was guided by many inquiries on the part of the friends of the method. I regret to say that I have not always been able to answer these questions as fully as I wished. I hope that my correspondents will find the desired explanation in this new version of the old essay. I deem it my duty, however, to say, in justice to Mr. Grube, that the following pages are not in every respect a translation from his work, as has been supposed by some. One gentleman has done me the credit to publish my essay over his own signature as a translation from Mr. Grube's work. It should be distinctly understood that the full credit for one and every idea contained herein belongs to

Mr. Grube, but that he is not responsible at all for the many imperfections in the manner in which his thoughts are stated here. In a few instances only, the writer has allowed himself to depart from Mr. Grube's ideas. The two essays are, may I be allowed to repeat, not altogether a translation, but rather an attempt to give a condensed account of the 160 pages of Mr. Grube's work.

The second essay was read before the St. Louis Normal School Association in 1876, when it appeared proper to supply the continuation of the course recommended by a method which had attracted the attention of many thinking educators of the land, from California (See San Francisco Report of 1872) to New Hampshire (See State Report of 1876). The second essay contains a recapitulation and continuation of the first essay. It presumes as little as its predecessor to *recommend*, but simply submits a new and important method to the thoughtful consideration of those who are interested in the matter. If circumstances permit, this little book will be followed by a text-book of Primary Arithmetic, based on Grube's Method.

L. S.

St. Louis, November, 1878.

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GRUBE'S METHOD

OF

TEACHING PRIMARY NUMBERS.

The old, long-established method in arithmetic is calculated to teach the first four processes of addition, subtraction, multiplication, division, in the order in which they are named, finishing addition with small and large numbers, before subtraction is begun, and so forth. A more recent improvement on this method consisted in excluding the larger numbers altogether at the beginning, and dividing the numbers on which the first four processes were taught, into classes, or so-called circles. The child learns each of the four processes with the small numbers of the first circle (i.e., from 1 to 10) before larger numbers are considered; then the same processes are taught with the numbers of the second circle, from 10 to 100, then of the third, from 100 to 1,000, and so forth.

Grube, however, went beyond this principle of classification. He discarded the use of large numbers, hundreds and thousands, at the beginning of the course, as others had done before him; but instead of dividing the primary work in arithmetic into three or four circles or parts only, i.e., from 1 to 10, 10 to 100, etc., he considered each number as a circle or part by itself, and taught it by a method that is to be set forth in the

following pages. He recommended that the child should learn each of the smaller numbers in succession, and all the operations within the range of each number, before proceeding to the next higher one, addition, subtraction, multiplication, and division, before proceeding to the consideration of the next higher number.

In order to guard against a mistake which has been made rather frequently, it should be stated that such examples only are considered to be within the limit of a number, and are to be taught in connection with it, in which a larger number than the one that is being considered does not appear in any way whatsoever. Thus, for instance, when the number four is taught, the teacher should exclude at the beginning addition and subtraction by fours, multiplication with 4 as one of the factors, division with 4 as the divisor, because these belong to a later and more advanced part of the course, since they involve in the sum, minuend, product, or dividend numbers beyond the limit of the one that is being considered. But all the examples that do not involve a higher number than four, are illustrated and taught, before passing over to the next higher number, five. Treating, for instance, the number 2, Grube leads the child to perform all the operations that are possible within the limits of this number, i.e., all those that do not presuppose the knowledge of any higher number, no matter whether in the usual classification these operations are called addition, subtraction, multiplication, or division. The child has to see and to keep in mind that

$$1 + 1 = 2, \quad 2 \times 1 = 2, \quad 2 - 1 = 1, \quad 2 \div 1 = 2, \text{ etc.}$$

The whole circle of operations up to 2 is exhausted

before the child proceeds to the consideration of the number 3, which is to be treated in the same way.

Why adhere to the abstract division of the work in arithmetic into addition, subtraction, etc., in the primary grade, where these distinctions do not help to make the subject any clearer to the pupil? The first four processes are naturally connected, and will appear so in the untaught mind. If you take away 1 from 2, and 1 remains, the child, in knowing this, also understands implicitly the opposite process of adding 1 to 1 and its result.

Multiplication and division are, in the same way, nothing but another way for adding and subtracting, so that we might say one operation contains all the others. "Every text-book of primary arithmetic professes to teach the numbers in some way or other," says Grube; "but to know a number really means to know also its most simple relations to those numbers, at least, which are smaller than it." Any child, however, who knows a number and its relations, must be also able to perform the operations of adding, subtracting, etc., for they are nothing but the expression of the relation in which one number stands to others. Each example shows what must be added to or subtracted from a number to raise it or lower it to equality with another, or, as in multiplication and division, it sets forth the multiple relation of two numbers.

The four processes are the direct result of comparing, or "measuring," as Grube calls it, two numbers with each other. Only when the child can perform all these operations, for instance, within the limits of 2, can it be supposed really to have a perfect knowledge of this number. So Grube takes up one number after the

other, and compares it with the preceding ones, in all imaginable ways, by means of addition, subtraction, multiplication, and division. This comparing or "measuring" takes place always on external, visible objects, so that the pupil can see the objects, the numbers of which he has to compare with each other. The adherents of this method claim for it that it is based on a sound philosophical theory, and that it has proved superior in practice to the methods in use before its invention.

Some of the most important principles of this method of instruction are given by Grube in the following:

"1 (Language). We cannot impress too much upon the teacher's mind, that each lesson in arithmetic must be a lesson in language at the same time. This requirement is indispensable with our method. As the pupil in the primary grade should be generally held to answer in complete sentences, loud, distinctly, and with clear articulation, so especially in arithmetic, the teacher has to insist on fluency, smoothness, and neatness of expression, and should lay special stress upon the *process* of solution of each example. As long as the language for the number is not perfect, the idea of the number is defective as well. An example is not finished when the result has been found, but when it has been solved in a proper way. Language is the only test by which the teacher can ascertain whether the pupils have perfectly mastered any step or not.

"2 (Questions). Teachers should avoid asking too many questions. Such questions, moreover, as, by containing half the answer, prompt the scholar, should be omitted. The scholar must speak himself as much as possible.

"3 (Class and Individual Recitation). In order to animate the lesson, answers should be given alternately by the scholars individually, and by the class in concert. The typical numeri-

cal diagrams (which, in the following, will continually re-appear) are especially fit to be recited in concert.

"4 (Illustrations). Every process and each example should be illustrated by means of objects. Fingers, lines, or any other objects will answer the purpose, but objects of some kind must always be presented to the class.

"5 (Comparing and Measuring). The operation of each new stage consists in comparing or measuring each new number with the preceding ones. Since this measuring can take place either in relation to difference (arithmetical ratio), or in relation to quotient (geometrical ratio), it will be found to comprise the first four rules. A comparison of two numbers can only take place by means of one of the four processes. This comparison of the two numbers, illustrated by objects, should be followed by exercises in the rapid solving of problems and a view of the numerical relations of the numbers just treated, in more difficult combinations. The latter offer a good test as to whether the results of the examination of the arithmetical relations of the number treated have been converted into ideas by a process of mental assimilation. In connection with this, a sufficient number of examples in applied numbers are given to show that applied numbers hold the same relation to each other that pure numbers do.

"6 (Writing of Figures). On neatness in writing the figures, the requisite time must be spent. Since an invariable diagram for each number will re-appear in all stages of this course of instruction, the pupils will soon become able to prepare the work for each coming number by writing its diagram on their slates."

It will appear from this that Mr. Grube subjects each number to the following processes:

I. Exercises on the pure number, always using objects for illustration.

a. Measuring (comparing) the number with each of the preceding ones, commencing with 1, in regard to addition, multiplication, subtraction, and division, each number being compared by all these processes before the next number is taken up for comparison. For instance, 6 is first compared with 1 by means of addition, multiplication, subtraction, and division,

$$(1 + 1 +, \text{ etc.} = 6;\ 6 \times 1 = 6;$$
$$6 - 1 - 1, \text{ etc.} = 1;\ 6 \div 1 = 6)$$

then with 2, then with 3, and so forth.

b. Practice in solving the foregoing examples rapidly.

c. Finding and solving combinations of the foregoing examples.

II. Exercises on examples with applied numbers.

In the following, Mr. Grube gives but the outline, the skeleton as it were, of his method, trusting that the teacher will supply the rest. The sign of division, as will be explained below, should be read at the beginning: "From . . . I can take away . . . — times." By this way of reading, the connection between subtraction and division becomes evident.

FIRST STEP.

THE NUMBER ONE.

"As arithmetic consists in reciprocal 'measuring' (comparing), it cannot commence with the number 1, as there is nothing to measure it with, except itself as the absolute measure."

I. The abstract (pure) number.

One finger, *one* line; one is once one.

The scholars learn to write:

● 1.

● 1 × 1 = 1.

II. The applied number.

What is to be found *once*, in the room, at home, on the human body?

SECOND STEP.

THE NUMBER TWO.

I. The pure number.

a. MEASURING (comparing).

● ● 2.

● 1 | 1 + 1 = 2.
2 × 1 = 2.

● 1 | 2 − 1 = 1.
2 ÷ 1 = 2. (Read: From 2 I can take away 1 twice.)

2 is one more than 1.

1 is one less than 2.

2 is the double of 1, or twice 1.

1 is one-half of 2.

b. PRACTICE IN SOLVING EXAMPLES RAPIDLY. $1 + 1 = ?$ $2 - 1 = ?$ $2 \div 1 = ?$ $1 + 1 - 1 \times 2 = ?$ etc.

c. COMBINATIONS.

What number is contained twice in 2?

2 is the double of what number?

Of what number is 1 one-half?

Which number must I double to get 2?

I know a number that has in it one more than one. Which is it?

What number have I to add to 1 in order to get 2?

II. Applied numbers.

Fred had two dimes, and bought cherries for one dime. How many dimes had he left?

A slate-pencil costs 1 cent. How much will two slate-pencils cost?

Charles had a marble, and his sister had twice as many. How many did she have?

How many one-cent stamps can you buy for 2 cents?

THIRD STEP.

THE NUMBER THREE.

I. The pure number.

a. MEASURING.

(1) By 1.

● ● ● 3.

● 1 $1 + 1 + 1 = 3.$

$3 \times 1 = 3.$

● 1 $3 - 1 - 1 = 1.$ (Better than $3 - 1 - 1 - 1 = 0$.)

for, $3 - 1 = 2$; $2 - 1 = 1$.

● 1 $3 \div 1 = 3.$

This ought to be read: From 3 I can take away 1 3 times, or, in three, 1 is contained three times. The ideas of "to be taken away" and "to be contained" must always precede the higher and more difficult conception of dividing.

(2) Measuring by 2.

$2 + 1 = 3;\ 1 + 2 = 3.$
$1 \times 2 + 1 = 3.$
$3 - 2 = 1;\ 3 - 1 = 2.$
$3 \div 2 = 1$ (1 remainder).

(From 3, I can take away 2 once, and 1 will remain; or, In three, 2 is contained once and one over.)

3 is 1 more than 2, 3 is 2 more than 1.

2 is 1 less than 3, 2 is 1 more than 1.

1 is 2 less than 3, 1 is 1 less than 2.

3 is three times 1.

1 is the third part of 3.

1 and 1 are equal numbers, 1 and 2, as well as 2 and 3 are unequal.

Of what equal or what unequal numbers does 3 consist, therefore? etc.

b. PRACTICE IN SOLVING EXAMPLES RAPIDLY.

How many are $3 - 1 - 1 + 2$ divided by 1?

$1 + 1 + 1 - 2 + 1 + 1 - 2 + 1 + 1 = ?$

$3 \times 1 - 2 \times 1 + 1 + 1 - 2 + 1 + 1 = ?$ etc.

The answers must be given immediately.

No mistakes can arise as to the meaning of these examples; the question whether $3 \times 1 - 2$ means $(3 \times 1) - 2$ or $3 \times (1 - 2)$ is answered by the fact that these examples represent oral work, and that it is supposed that the operation indicated by the first two numbers (3×1) is completed mentally before the next number is given.

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c. Combinations.

From what number can you take twice 1 and still keep 1?

What number is three times 1?

I put down a number once, and again, and again once, and get 3; what number did I put down 3 times?

II. Applied numbers.

How many cents must you have to buy a three-cent stamp?

Annie had to get a pound of tea for 2 dollars. Her mother gave her 3 dollars. How much money must Annie bring back?

Charles read one line in his primer, his sister read 2 lines more than he did. How many lines did she read?

If one slate-pencil costs one cent, how much will 3 slate-pencils cost?

Bertha found in her garden 3 violets, and took them to her parents. How can she divide them between father and mother?

FOURTH STEP.

THE NUMBER FOUR.

I. The pure number.

a. Measuring.

(1) By 1.

● ● ● ● 4.

●	1	$1 + 1 + 1 + 1 = 4$ $(1 + 1 = 2,\ 2 + 1 = 3)$.
●	1	$4 \times 1 = 4$.
●	1	$4 - 1 - 1 - 1 = 1$.
●	1	$4 \div 1 = 4$.

(2) Measuring by 2.

2 | 2 + 2 = 4.
2 × 2 = 4.
4 − 2 = 2.
4 ÷ 2 = 2.

(3) Measuring by 3.

3 | 3 + 1 = 4, 1 + 3 = 4.
1 × 3 + 1 = 4.
1 | 4 − 3 = 1, 4 − 1 = 3.
4 ÷ 3 = 1 (1 remainder).

(In 4, 3 is contained once and 1 over; or from 4 I can take away 3 once, and one remains.)

Name animals with 4 legs and with 2 legs.

Wagons and vehicles with 1 wheel, 2, and 4 wheels. Compare them.

4 is 1 more than 3, 2 more than 2, 3 more than 1.
3 is 1 less than 4, 1 more than 2, 2 more than 1.
2 is 2 less than 4, 1 less than 3, 1 more than 1.
1 is 3 less than 4, 2 less than 3, 1 less than 2.
4 is 4 times 1, twice 2.
1 is the fourth part of 4, 2 one-half of 4.

Of what equal and unequal numbers can we form the number 4?

b. PROBLEMS FOR RAPID SOLUTION.

2 × 2 − 3 + 2 × 1 − 1 − 2 + 2 = ?
4 − 1 − 1 + 1 + 1 − 3, how many less than 4? etc.

c. COMBINATIONS.

What number must I double to get 4?
Four is twice what number?
Of what number is 2 one-half?
Of what number is 1 the fourth part?
What number can be taken twice from 4?

What number is 3 more than 1?
How much have I to add to the half of 4 to get 4?
Half of 4 is how many times one less than 3? etc.

II. Applied numbers.

Caroline had 4 pinks in her flower-pot, which she neglected very much. For this reason, one day one of the flowers had withered, the second day another, and the following day one more. How many flowers did Caroline keep?

How many dollars are 2 + 2 dollars?

Three apples and one apple?

4 quarts = 1 gallon.

Annie bought a gallon of milk; how many quarts did she have?

She paid 1 dime for the quart; how many dimes did she pay for the gallon?

4 { ● quart, ● quart, ● quart, ● quart,

4 { ● dime. ● dime. ● dime. ● dime.

What part of 1 gallon is 1 quart?

If 1 quart costs 2 dimes, can you get a gallon for 4 dimes?

A cook used a gallon of milk in 4 days. How much did she use each day?

The recitations should be made interesting and animated by frequently varying the mode of illustration, and in this the ingenuity of the teacher and her inventive power can display themselves to their best advantage. It is, of course, superfluous to describe the infinite variety of objects which may be used,

but a few suggestions will perhaps prove acceptable. Those illustrations which compel the whole class to be active; or which are of special interest, and arouse the attention of pupils, are of greater value than others. For instance:

"Class, raise two fingers of your right hand; two fingers of your left hand. How many fingers have you raised? Two fingers and two fingers are how many? Two and two are how many? Carrie may show to the class, with her fingers, that two and two are four."

This plan of illustrating should be used very frequently, as it requires the whole class to be active. The following illustration is also commendable, as it hardly ever fails to enlist the interest of the class; every pupil likes to be allowed to illustrate a problem in this way:

"From four I can take away two, how many times? Emma may show that her answer is correct, by making some of the other girls stand." (The class know that those whom Emma teaches must stand until she makes them take their seats again.) Emma: "Four little girls are standing here. From 4 little girls I can take away 2 little girls once (making two of the four take their seats), twice (making the other two sit down). From 4 little girls, I can take away 2 little girls twice. From 4 I can take away 2 twice. $4 \div 2 = 2$."

FIFTH STEP.

THE NUMBER FIVE.

I. **The pure number.**

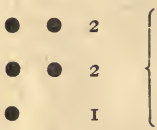
a. MEASURING.

(1) By 1.

● ● ● ● ● 5.

$1 + 1 + 1 + 1 + 1 = 5.$
$5 \times 1 = 5.$
$5 - 1 - 1 - 1 - 1 = 1.$
$5 \div 1 = 5.$

(2) By 2.



$2 + 2 + 1 = 5.$
$2 \times 2 + 1 = 5.$
$5 - 2 - 2 = 1.$
$5 \div 2 = 2$ (1 remainder).

(3) By 3.

$3 + 2 = 5$; $2 + 3 = 5.$
$1 \times 3 + 2 = 5.$
$5 - 3 = 2$; $5 - 2 = 3.$
$5 \div 3 = 1$ (2 remainder).

(4) By 4.

$4 + 1 = 5$; $1 + 4 = 5.$
$1 \times 4 + 1 = 5.$
$5 - 4 = 1$; $5 - 1 = 4.$
$5 \div 4 = 1$ (1 remainder).

The fingers are the best means of illustration here: "Hold up your left hand. How many fingers are you holding up?

Hold the thumb away from the other fingers. How many fingers here? (1); here? (4). 1 finger taken from 5 fingers leaves how many fingers? 1 from 5 = ? 4 fingers + 1 finger = ? 4 + 1 = ? Hold your first finger and the thumb away from the other fingers. 5 − 2 = ? 3 + 2 = ? 2 + 3 = ?" etc.

5 is one more than 4, 5 is 2 more than 3, 5 is 3 more than 2, 5 is 4 more than 1. (All the solutions of these examples are the result of observation from illustrations placed before the eyes of the class; without them this kind of instruction is worthless.)

4 is 1 less than 5; 4 is 1 more than 3, etc.

3 is 2 less than 5, etc.

5 = 5 × 1.

1 = $\frac{1}{5}$ × 5 (1 is the fifth part of five).

5 consists of two unequal numbers, 3 + 2. 5 consists of two equal numbers and one unequal number, 2 + 2 + 1.

b. PRACTICE IN THE RAPID SOLUTION OF EXAMPLES.

(It would be a great mistake to drill on the same example until the pupils can remember it. Such a practice would be worse than valueless; every example should be a new one to the pupil, and the faculty appealed to should be judgment as well as memory.)

5 − 2 − 3 + 2 × 2, one-half of it less 1, taken 5 times = ?

2 × 2 + 1 − 3 × 1 × 2 − 3 + 4 = ? etc.

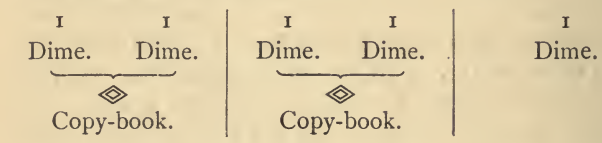
c. COMBINATIONS.

What number is one fifth of 5? How many must I add to 3 to get 5? How many must be taken away from 5 to get 3? How many times two have I added to 1 in order to get 5? I have taken away twice 2 from a certain number, and 1 remained. What number was it? etc.

II. **Applied numbers.**

How many gallons are 2 quarts?

Charles had 5 dimes; he bought two copy-books, each of which cost two dimes. What money did he keep? (This the teacher must make plain by means of lines and dots.)



Henry read a lesson three times, Emma read it as many times as he did, and two times more. How often did she read it? Father had five peaches, and gave them to his 3 children. The youngest one received one peach; how many did each of the other children receive? etc.

SIXTH STEP.

THE NUMBER SIX.

I. **The pure number.**

a. Measuring.

aa with 1
bb with 2
cc with 3
dd with 4
ee with 5

Each process illustrated by six lines, of which as many are placed in a row as are indicated by the number by which 6 is to be measured.

ff Miscellaneous examples.

b. Rapid solution of problems.

c. Combinations of numbers.

II. **The applied number.**

Grube thinks that one year ought to be spent in this way on the numbers from 1 to 10. He says, "In the thorough way in which I want arithmetic taught, one year is not too long for this most important part of the work. In regard to extent, the scholar has not, apparently, gained very much — he knows only the numbers from 1 to 10. But he knows them."

In reference to the main principles to be observed, he demands, first, "that no new number shall be commenced before the previous one is perfectly mastered;" secondly, "that reviews should frequently and regularly take place;" and lastly, "that whatever knowledge has been acquired and fully mastered by illustration and observation, must be thoroughly committed to memory." "In the process of *measuring*, pupils must acquire the utmost mechanical skill." It is essential to this method, that in the measuring, which forms the basis for all subsequent operations, the pupils have before their eyes a diagram illustrating the process. It matters not by means of what objects the pupils see the operation illustrated, whether fingers, lines, or dots, but they certainly must see it.

It is a feature of this method, that it teaches by the eye as well as by the ear, while in most other methods arithmetic is taught by the ear alone. If, for instance, the child is to measure 7 by the number 3, the illustration to be used is:

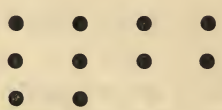
If lines or dots are arranged in this way, and impressed upon the child's memory as depicting the rela-

tion between the numbers 3 and 7, it is, in fact, all there is to know about it. Instead of teaching all the variety of possible combinations between 3 and 7, it is sufficient to make the child keep in mind the above picture. The first four rules, as far as 3 and 7 are concerned, are contained in it, and will result from expressing the same thing in different words, or describing the picture in different ways. Looking at the picture, the child can describe it, or read it as:

$3 + 3 + 1 = 7$, or $2 \times 3 + 1 = 7$,
or $7 - 3 - 3 = 1$; $7 \div 3 = 2$ (1).

The latter process to be read, From 7 I take away 3 twice, and 1 remains; or, 7 contains 3 twice and one more.

Let the number to be measured be 10, and the number by which it is to be measured be 4; then since the way to arrange the lines or dots for illustration is to have as many dots or lines as are indicated by the larger number, and as many of them in a row as are indicated by the smaller number, we write:



The child will be able to see at once, by reading the diagram, as it were, that

$4 + 4 + 2 = 10$; $2 \times 4 + 2 = 10$;
$10 - 4 - 4 = 2$; $10 \div 4 = 2$ (2),

and to perceive at a glance a variety of other combinations. The children will, in the course of time, learn

how to draw these pictures on their slates in the proper way. Nor will it take long to make them understand that every picture of this kind is to be "read" in four ways, first using the word *and*, then *times*, then *less*, then, *From . . . can be taken away . . . times.* As soon as the pupils can do this, they have mastered the method, and can work independently all the problems, within the given number, which are required in measuring.

It would be a mistake to suppose that, in teaching according to this method, memory is not required on the part of the child. Memory is as important a factor here as it is in all instruction. This should be emphasized, because with some teachers it has become almost a crime to say that memory holds its place in education. To have a good memory, is, in their eyes, a sign of stupidity. Grube was too experienced a teacher to fall into this error. While by his method the results are gained in an easier and more natural way, whatever result is arrived at must be firmly retained by dint of memory assisted by frequent reviews.

(End of First Essay.)

NUMBERS ABOVE TEN.

Second Essay.

When Grube's Method of teaching the elements of arithmetic was first introduced to the Teachers' Association of St. Louis, in 1870, it was not presented with the assurance of warranted success as the only plan of teaching this important study, but rather as an attempt to demonstrate practically, in a given instance, to some extent at least, how far methods of teaching may be redeemed from the bane of vagueness, which, as long as it lasts, excludes them from the rank, in the science of Pedagogics, to which they might otherwise be entitled. Grube's Method was submitted with diffidence to the judgment of practical teachers, without the commendation of any champion who expressed an implicit belief in its immediate signal success. We may speak disparagingly of the often frivolous distinction between theory and practice, which ignores the harmonious parallelism between the world of thought and the world of fact, but we shall nevertheless insist upon practical usefulness as the test of any psychologically correct method of teaching.

To-day Grube's Method of teaching arithmetic does not lack friends and supporters: it has been tried and adopted, not in one city alone, but has become recognized throughout the country. Long before its practi

cal test in the district schools of St. Louis, it was made part of the regular course of instruction in the schools of San Francisco, and many other cities have adopted it since. Practical experience has shown the advantages and disadvantages of this system as far as the part which was presented at that time is concerned; namely, the numbers from one to ten only.

Beyond this limit there is still disputed ground, and it may be allowable to say that the continuation is offered to-day in the same spirit as the beginning was years ago. It is simply a report on an ingenious method which is considered worthy the notice of thoughtful teachers, and which seems to deserve a fair trial, continued for a sufficient length of time to extend beyond the period during which a new method seems objectionable because it is new, and hence collides with a practice which habit has made convenient.

The leading idea is the same throughout Grube's Method. To show the principle of teaching the higher numbers to 100 is to recapitulate the principles that are to guide the teacher in his treatment of the numbers from 1 to 10. That the four processes are taught with each number, before the following one is considered, forms, no doubt, a characteristic feature of Grube's Method, but it is a common mistake to suppose that it is the leading idea. It certainly emanates from this idea, but it is not the idea itself. The leading principle is rather that of objective illustration.

In a very general way it may be said that in examples in primary arithmetic two numbers are given, and their relation, expressed by a third number, is to be found. Hence the elementary processes may be considered as the comparing of one number with the other, or the

measuring of one by the other. On the basis of this general theory, Grube suggests a general plan of illustration, according to which the larger number of the two numbers given is represented by the total number of lines or dots placed on the blackboard. These lines are arranged into sets or groups, each containing as many lines or dots as are indicated by the smaller number of the two. Thus, if the numbers 6 and 2 are to be compared with each other, the illustration consists of six dots, arranged two by two.

The measuring of 9 by 4 is illustrated by four dots and four dots and one dot.

This contains the main principle of Grube's Method. If perception has seized this illustration, and wrought it into a mental picture, the solution of all the existing elementary relations between the two numbers has been grasped implicitly. For the four processes are simply different interpretations of this symbolic diagram. When this picture appears before the mind, it may be interpreted as addition or multiplication, i.e., our illustration may be read,

$$4 + 4 + 1 = 9, \text{ or } (2 \times 4) + 1 = 9;$$

and by the retrograde process, when the illustration is made to disappear from the blackboard, it may be in-

terpreted by subtraction or division, as $9 - 4 - 4 = 1$, or from 9 I can take away 4 twice and leave 1. But the main point in this is, that the whole process is based on well selected and arranged illustrations, and is an object lesson on numbers. A plan of teaching which ignores this main point, and flatters itself to have found the gist of the new idea by jumbling together addition, subtraction, multiplication, and division, without the most extensive use of illustrative objects, and without systematic arrangement, has nothing in common with Grube's Method. In the latter, the clearest order and regularity prevail throughout. Below 10, each number is compared with the number 1, by means of addition, subtraction, multiplication, and division, then with the number 2, then with 3, etc. The pupil will soon learn to perceive the regularity of this process; and at the moment he has understood that part, he can by independent work discover the primary arithmetical relations of a number, and prepare a synopsis or diagram of the same.

A frequent and very dangerous mistake is the omission, or neglect, of applied examples. The pure number as the universal expression of arithmetical truth is of the greatest importance, but the pupil throughout his school-course finds the greatest difficulty in working with applied numbers. Moreover, arithmetic is studied for life; and in life, there are none but applied examples. Hence, after the universal, the pure number, has been mastered by means of observation, particular application should follow immediately, and copious examples, clothed in the most varied forms, should be solved. The training which the pupil receives from practice with applied problems is different in kind from that with pure

numbers, and hence cannot be slighted in the primary grades without retarding the progress in the higher classes. Without sufficient practice in this direction, there is danger of mechanical and dull work, and the best opportunities for the pupil's display of inventive ingenuity are lost.

The difficulty which the study of arithmetic presents in the higher grades lies not in the mechanical handling of numbers, — in most cases the pupils succeed very well in that, — but it lies in the fact that the words of the problem puzzle them. The qualitative element disturbs and conceals the quantitative. If this assertion is correct, a great deal of training with applied numbers should be given at a time of the course when the pure number which is considered is so small as to allow the scholar, after having mastered it, to concentrate his whole attention on the puzzle that lies in the wording, in the qualitative. Wherever sufficient training of this kind has sharpened the wit of the pupils in the lower grades, they will no longer consult the heading of the chapter as the first step in the solution of a problem, in order to find whether it means addition or division, interest or long measure, and find themselves in a helpless and forlorn condition when they meet an example which is not labeled by any heading.

An analysis of the operation with each number shows as the two principal elements :

I. *The number considered in its universal quantitative character, or pure number.* The process is from objectivity to abstraction.

II. *The quantitative in special qualitative form, or applied number.* Here we proceed from abstraction to application.

Under the first or pure number we have the sub-topics:

a. Comparing with, or measuring by, each of the preceding numbers, from 1 to 10, considering addition, multiplication, subtraction, and division.

b. Combinations of the two numbers treated of, the results to be within the limits of the greater one of the two numbers. This is a very important process, no doubt, but the temptation lies near to give too much prominence to it by forgetting that it is a part only of Grube's Method. The systematic comparison of numbers is of greater importance, and it is an error to spend as much time on these combinations as if the method consisted of nothing but these.

c. Sufficient practice in the rapid solution of examples.

In the former essay, the treatment of the numbers from one to five was explained. As the last step within the circle of numbers from one to ten, and as the transition to the province of larger numbers, the treatment of the number ten is of great importance. Grube describes it in the following way:

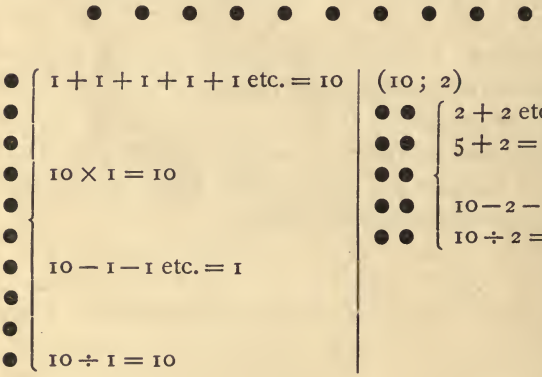
TENTH STEP.

THE NUMBER TEN.

We have arrived at a number which is again treated as a unit. Hence we write it by means of the figure one; but to show there is ten times as much in this as in the figure one which we had before, we move it one place toward the left, by which we mean to say, This unit means a ten. The empty place of the simple unit is filled out by a cipher.

I. The pure number.

a. MEASURING (10; 1).



Miscellaneous Measuring:

10 consists of two equal numbers, $5 + 5$.

10 consists of five equal numbers, $2 + 2 + 2 + 2 + 2$.

10 consists of two equal numbers and one unequal, $3 \times 3 + 1$.

10 consists of four unequal numbers, $1 + 2 + 3 + 4$.

Review of the multiple relations within the number ten.

A. I. 1 is one-half of 2, one-third of 3, one-fourth of 4, etc.
II. 2 is one-half of 4, one-third of 6, etc.
III. 3 is one-half of 6, one-third of 9.
IV. 4 is one-half of 8.
V. 5 is one-half of 10.

B. I. 10 is 10 times 1, 5 times 2, 2 times 5.
II. 9 is 9 times 1, 3 times 3.
III. 8 is 8 times 1, 4 times 2, 2 times 4.
IV. 7 is 7 times 1.
V. 6 is 6 times 1, 3 times 2, 2 times 3.
VI. 5 is 5 times 1.
VII. 4 is 4 times 1, 2 times 2.
VIII. 3 is 3 times 1.
IX. 2 is 2 times 1.
X. 1 is once 1.

What numbers are contained without any remainder in 10, 9, 8?

What numbers have no other numbers contained in them without remainder except the number 1? (The prime numbers 1, 3, 5, 7.)

b. Combinations (Oral work).

One nickel and two cents and three cents, less 6 cents, of this take one-half three times, and add twice two cents; how many cents?

(There is no better exercise for rapidity and exactness than this. Short combinations, slowly pronounced at first, until the class can solve more difficult problems, given out quickly. The teacher should take care not to discourage the class by examples that can be answered by the brightest scholars only. No guessing should be allowed; use illustrations.)

$(2 \times 2) + (2 \times 3) - (3 \times 3) + (2 \times 4) + 1 = ?$
$10 - 2 - 1 - 2 - 1 - 2 - 1 = ?$
$1 + 2 + 3 + 4 = ?$ etc.

c. PRACTICE IN THE RAPID SOLUTION OF EXAMPLES.

What number is 1 more than twice 3?

Twice five is how many more than three times three? than twice four?

A father distributed 10 apples among his children, so that each older child received one more than the one next below him in age. How many apples did each child receive? (The pupils know that 10 consists of 4 unequal numbers, 1, 2, 3, 4, of which each following number is greater by one than the preceding. Hence the father could divide the apples so that the youngest received one, the next two, etc.)

Charles had learned four words in spelling. His brother said, "I know twice as many as you, and 2 more." How many did he know? Solution: If Charles had learned 4 words, and his brother knew twice as many and 2 more, he knew $2 \times 4 + 2$ words $= 10$ words.

William said, "I am 5 times as old as my little sister." She was 2 years old. How old was William? Solution: If the little sister was 2 years old, and William five times as old, he was 5×2, or 10 years old.

II. Applied numbers.

10 days are how many weeks and days?

10 cents are how many dimes? nickels?

Fred had one dime: he bought 2 slate-pencils for one cent each, and a piece of candy for 5 cents. How much money did he spend? How much had he left?

One lead-pencil costs 5 cents; how much will two lead-pencils cost?

How many marbles at 2 cents apiece can you buy for 10 cents? Solution: For 2 cents I get 1 marble, hence for 10 cents I get five marbles, since 10 cents are 5×2 cents. Or, If I give to the store-keeper 2 cents, he gives me 1 marble; but if I have 10 cents, I can give him 5 times 2 cents, and so he gives me 5 times one marble, etc.

With this number, says Grube, the first and most important step in arithmetic has been completed. If the subject has been taught as it should have been, one year is not too long a time for it. (It seems that Grube is speaking on the supposition that about two hours a week form the time given to the study of arithmetic.) The pupil's knowledge is not very extensive: he knows but the numbers from 1 to 10. But would he really possess any knowledge of arithmetic if he were able to count up to 100 and beyond without being able to solve any problem, even with the smallest number? Learning the names of numbers up to 100 is not the same as learning to count, and is simply learning by rote a series of words, not of much more importance for arithmetic than the committing to memory of a few lines of poetry. Special attention should be given to the practice in the rapid solution of miscellaneous examples (I. *c.*), as these exercises are essential for a clear idea of number. If clear perception has preceded them, they will present no difficulty. Remember that no new number is to be taken up before the previous one has

been thoroughly mastered, and that frequent reviews must help the pupil in fixing in his memory the principal examples which have been considered and reduced to writing.

Passing over to the higher numbers, Grube says, "In the second year the numbers from 10 to 100 are to be studied. The following principles must be observed in this work:"

1. Fingers and lines are used for illustration, the former being the most natural means.

2. The process with the numbers from 10 to 100 is the same as that for the smaller numbers. Multiplication and division form the subjects of written and oral work, while addition and subtraction, as a rule, need oral treatment only. Measuring each new number by the numbers from 1 to 10 is continued as oral, preparatory work until the pupils have acquired in it the greatest mechanical skill.

3. Greatest diversity of expression and sufficient variety are aimed at in the selection of examples, in pure as well as in applied numbers, so that the pupil may free himself gradually from the uniformity of the elementary diagram and schedule. Applied examples should not go beyond the limit of qualitative relations taken from daily life with which the pupil is familiar. This will give him an opportunity of inventing examples himself, and the permission to give an example to the class may be made a reward for that pupil who succeeds in finding the solution of some examples first.

Before proceeding to describe Grube's treatment of some numbers of the circle from 10 to 100, it will be best to recall to memory the few essential points of

difference and agreement with the previous part of the course.

1. The processes with each number remain the same, namely:

1. *Exercises with the pure number*, by
   (*a*) Comparison.
   (*b*) Combination.
   (*c*) Practice in the rapid solution of examples.
2. *Exercises with applied number.*

2. Objective illustrations form the most important part of each exercise. Arithmetic is a series of object lessons on numbers.

3. Each new number is not compared with all the numbers below itself, but with the numbers from 1 to 10 only.

4. Comparison with these numbers by means of addition and subtraction forms as a rule the subject of oral work only: comparison by multiplication and division is practised both orally and in writing.

5. In writing out these comparisons of numbers, the examples are no longer placed side by side, but below each other:

(11; 2)
$$\begin{aligned} 2 + 2 &= 4 \\ 4 + 2 &= 6 \\ 6 + 2 &= 8 \\ 8 + 2 &= 10 \\ 10 + 1 &= 11 \end{aligned}$$

6. Oral comparison by addition and subtraction takes usually the form of, Count upward or downward by twos, threes, fours, etc.

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7. As the same examples occur frequently, Grube supposes that the pupil has acquired sufficient skill to master about two numbers each recitation; he is speaking, however, of recitations of 60 minutes each.

8. More time is to be given to the lower numbers from 1 to 24, and especially to numbers that are of importance in applied examples as representing some division in compound numbers, such as 12 (dozen, number of months, etc.), 14 (days in 2 weeks), 15, 16 (number of ounces in a pound), 18, 20, 21, 24, 25, 28, 30, 36, 48, 56, 64, 72, etc. In connection with them the principal divisions of compound numbers should be taught.

After this general explanation, an application of the principles set forth to a few particular numbers will suffice to show the process.

TWELFTH STEP.

THE NUMBER TWELVE.

I. *a.* — **Pure number.**

MEASURING.

10 + 2 = 12

Oral work. — Measuring.

(12 ; 1)	(12 ; 2)	(12 ; 3)	(12 ; 6)
1 + 1 =	2 + 2 =	3 + 3 =	6 + 6 = 12
2 + 1 =	4 + 2 =	6 + 3 =	
3 + 1 =	6 + 2 =	9 + 3 =	
11 + 1 =	etc.		
or	or		
1, 2, 3, 4, etc.	2, 4, 6, 8, 10, 12	or 3, 6, 9, 12	
12 − 1 =	12 − 2 =	12 − 3 =	12 − 6 =
11 − 1 =	10 − 2 =	9 − 3 =	
10 − 1 =			
9 − 1 =			
etc.			
or		or	
12, 11, 10, etc.	or 12, 10, 8, 6	12, 9, 6, 3	
12 × 1 = 12	6 × 2 = 12	4 × 3 =	2 × 6 =
12 ÷ 1 = 12	12 ÷ 2 =	12 ÷ 3 =	12 ÷ 6 =

WRITTEN WORK.

$12 = 12 \times 1$
$= 6 \times 2$
$= 4 \times 3$
$= 3 \times 4$
$= 2 \times 5 + 2$
$= 2 \times 6$
$= 1 \times 7 + 5$
$= 1 \times 8 + 4$
$= 1 \times 9 + 3$
$= 1 \times 10 + 2$
1 tenth 2 units

$12 = 12 \times ?$
$12 = 6 \times ?$
$12 = 4 \times ?$
$12 = 3 \times ?$
1) $12 = 12 \times$
2) $12 = 6 \times$ etc.

To be read:

a. 1 can be taken away from 12, 12 times.

b. 1 is contained or is in 12, 12 times.

c. 1 is the 12th part of 12, etc.

$12 = 11 + 1$
(12 is 1 more than 11.)
$= 10 + 2$
$= 9 + 3$

Of what equal numbers is twelve composed?

Of what unequal numbers?

Give three numbers that make twelve, of which each following number is two more than the previous one.

b. — COMBINATIONS. (Oral.)

$(2 \times 2) + (2 \times 2) + (2 + 2) = ?$

$2 + 3 + 3 + 2 + 2 - 4 + 4 - (4 \times 2) = ?$

Charles, Fred, and George had 12 apples; they ate one-half of them and one more; how many had they left? how many did they eat? etc.

c. — PRACTICE IN THE RAPID SOLUTION OF EXAMPLES.

The third part of 12 is what part of 8?

One-half of 12 is how many times 3?

What is the difference between one-half of 12 and one-half of 10?

12 is three times what number?

What number must I take from 12 to have 9?

What number taken away from 12 leaves 4? etc.

II. Applied number.

12 pieces equal a dozen. Half a dozen = ?

12 months are called a year. (The names of the months are to be committed to memory.)

What part of a dozen are six pieces?

What part of a year are six months?

3 months are a quarter (of a year).

3 pieces are a quarter of a dozen.

A month has about 4 weeks. Fred pays $12 a month for piano lessons; how much does he pay a week?

Solution: One month has 4 weeks. If he pays for 4 weeks $12, he pays for one week the fourth part of 12, which is $3.

A father paid $2 a month for private lessons given to his son. How much did he pay in a quarter? in half a year?

How many slate-pencils at three cents apiece can you buy for 12 cents?

Illustrate:

Caroline learned by heart 12 definitions in three days, etc. How many each day? etc.

The teacher should prepare collections of such examples in writing.

The numbers from 10 to 100 are treated in a similar way; as a further illustration, the treatment of the number 30 is given in full. Such numbers as 17, 19, 22, 23, 26, etc., which are of less importance than numbers that represent some frequently occurring division in the denomination of number (12, 18, 24, 36 = dozen, months, 7 = days, 10, 15 = cents, etc.), are treated in their relation as pure numbers only, and the processes taken up under II. are omitted with them.

THIRTIETH STEP.

THE NUMBER THIRTY.

(3 times the fingers of two hands.)

I. *a.* — 1. CONNECTION WITH FORMER STEPS: If I add one unit to 29 we have 3 tens.

Three tens are called thirty.

a. — 2. MEASURING BY THE NUMBERS FROM 1 TO TEN.

Oral.

(30; 1)	(30; 2)
Count from 1 to 30.	2, 4, 6, 8, 10, etc.
Count from 30 to 1.	30, 28, 26, 24, etc.

(30; 3) (30; 4) (30; 5)

(30; 6) (30; 10)

6, 12, 18, 24, 30

30, 24, 18, 12, 6

10 + 10 = 20
20 + 10 = 30
30 − 10 = 20
20 − 10 = 10

30 × 1 = 15 × 2 = 5 × 6 = 3 × 10 = 30
30 ÷ 1 = 30 ÷ 2 = 30 ÷ 6 = 30 ÷ 10 = 3

30 = 29 + 1
28 + 2
etc.

In counting by 2's, 3's, etc., a pupil should point to the illustration. The teacher should stop frequently in this exercise, and make the pupils state how many tenths and units they have counted so

far, and how many they have still to count up to 30. For instance: Class, 1, 2, 3, 4, 5, 6, 7, — "Stop." Pupil: "We are within the first ten, three more are necessary to complete the first ten, 23 units to make up 30." The same should be practised in counting downward.

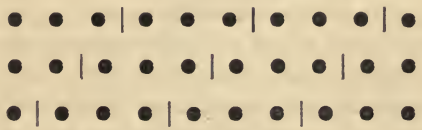
Written Work.

$30 = 30 \times 1$	$30 \div 1 = 30$*	$1 = \frac{1}{30} \times 30$ (is the 30th part of)
$= 15 \times 2$	$30 \div 2 = 15$	$2 = \frac{1}{15} \times 30$
$= 10 \times 3$	$\div 3 = 10$	
$= 7 \times 4 + 2$	$\div 4 = 7\ (2)$	
$= 6 \times 5$	$\div 5 = 6$	
$= 5 \times 6$	$\div 6 = 5$	
$= 4 \times 7 + 2$	$\div 7 = 4\ (2)$	
$= 3 \times 10$	$\div 10 = 3$	

30 is composed of what equal numbers?

30 is composed of which 2, 3, 4, etc., unequal numbers?

In these operations the 30 dots on the board should be separated into groups of 2, 3, etc., by placing lines between them, i.e. (30; 3)



If these suggestions meet with that support on the part of teachers which, as a rule, is most generously given to new

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* These examples are to be read by the pupil in several ways: *a*. From . . . I can take away . . . times. *b*. In 30 . . . is contained . . . times. *c*. The . . . th part of 30 is . . .

methods of value, it would be a good plan to have 10 lines or dots painted in a convenient place on the board in the rooms of the lowest grades, and 100 lines or dots arranged 10 by 10 painted on the board in the next higher rooms. The chalk-dots by which the pupils divide the lines or dots into groups might then be wiped off, when a new relation is taken up, without erasing the painted lines.

b. — COMBINATIONS WITHIN THE LIMITS OF 30.

1 dozen + 4 pieces + 2 pieces + $\frac{1}{2}$ dozen = ?

1 dime + 5 cents + 1 dime = how many cents?

$(3 \times 5) + (2 \times 4) + 7 - 15 - 8 + 5 + 9 = ?$

4×6, one-half, again one-half, 5 times = ? etc.

c. — EXERCISES IN RAPID CALCULATION.

Take 19 from 30 ($19 = 10 + 9$; $30 - 10 = 20$; $20 - 9 = 11$; $30 - 19 = 11$).

Twice 15 (15 = 1 ten and 5 units, 2×1 ten = etc.).

Compare 30 with 16 (30 = 3 tens; 16 = 1 ten and 6 units; 4 units must be added to the six to complete the second 10, and another ten to make it 3 tens. Hence 30 is 1 ten and 4 units, or 14 more than 16).

II. Applied examples.

(30 pieces = $2\frac{1}{2}$ dozen; 30 months = $2\frac{1}{2}$ years, etc.)

A great variety of these examples should be given; but even more important than this is the thoroughness with which each example is illustrated and worked through. Let the teacher move quietly in the stereotyped form of this method, so that the pupil becomes strong and self-active in the application of the familiar process. This apparently mechanical form rests on self-activity, and leads to self-reliance, self-confidence, and skill. More pupils fail in arithmetic from diffidence than from any other cause. In conclusion of the numbers of the second circle, the method of teaching the number 100 is given.

HUNDREDTH STEP.

NUMBER ONE HUNDRED.

Considerable time should be spent on the number 100. Besides the regular process which has been explained in connection with the number 30, a general review should take place. The multiplication table, of which the elements are known from previous instruction, may be written out in the following well-known forms, and committed to memory thoroughly.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

2× 2= 4									
2× 3= 6	3× 3= 9								
2× 4= 8	3× 4=12	4× 4=16							
2× 5=10	3× 5=15	4× 5=20	5× 5=25						
2× 6=12	3× 6=18	4× 6=24	5× 6=30	6× 6=36					
2× 7=14	3× 7=21	4× 7=28	5× 7=35	6× 7=42	7× 7=49				
2× 8=16	3× 8=24	4× 8=32	5× 8=40	6× 8=48	7× 8=56	8× 8=64			
2× 9=18	3× 9=27	4× 9=36	5× 9=45	6× 9=54	7× 9=63	8× 9=72	9× 9=81		
2×10=20	3×10=30	4×10=40	5×10=50	6×10=60	7×10=70	8×10=80	9×10=90	10×10=100	

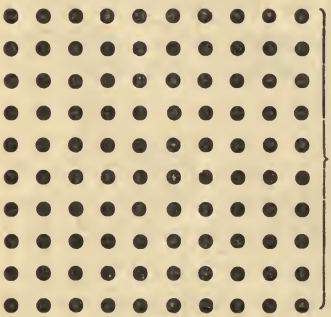
The pupil will understand from this that the product remains the same, no matter in what order the factors are multiplied.

MEASURING. — *Oral Work.*

I. *a.* Counting up to 100 and down by 2's, 3's, etc., to 10's, beginning with 1, 2, or any other number. The pupil must be able to do this without hesitation, and correctly, before this part of the course can be considered finished. The written form for exercises of the same kind is:

1 + 2 = 3	4 + 5 = 9	or, 1, 3, 5, 7, 9, etc.
3 + 2 = 5	9 + 5 = 14	
5 + 2 = 7	14 + 5 = 19	4, 9, 14, 19, etc.
etc.	etc.	

WRITTEN AND ORAL WORK.



10 × 10 = 100.

100 = 100 × 1	1) 100 = 100
= 50 × 2	2) 100 = 50 *
= 33 × 3 (+ 1)	3) 100 = 33 (1)
= 25 × 4	4) 100 = 25
= 20 × 5	5) 100 = 20
= 16 × 6 (+ 4)	6) 100 = 16 (4)

* To be read: *a.* 2 is contained in 100 fifty times. *b.* Half of 100 is 50, etc.

$100 = 14 \times 7\ (+2)$
$= 12 \times 8\ (+4)$
$= 11 \times 9\ (+1)$
$= 10 \times 10$
$= 99 + 1$
$= 98 + 2$ etc.

$7)\ 100 = 14\ (2)$
$8)\ 100 = 12\ (4)$
$9)\ 100 = 11\ (1)$
$10)\ 100 = 10$

I. *b.* Miscellaneous exercises in addition, subtraction, multiplication, and division, with all the numbers below 100, will be a test whether the pupil has the necessary mechanical skill to proceed to the study of numbers above 100.

Examples like the following should offer no difficulty.

ADDITION: $14 + 13 + 12 + 11$
$15 + 17 + 19 + 18$
$25 + 37 + 39 + 17$

SUBTRACTION: $90 - 16 - 12 - 11$
$98 - 32 - 41 - 24$

MULTIPLICATION: 3×30, 4×22, 2×44, 2×27.
3×25, 35×2, etc.

DIVISION: $3)\ 60$, $3)\ 69$.
$4)\ 60$, $4)\ 72$.
$12)\ 84$, $13)\ 65$.
$4)\ 53$.

A good exercise in the combination of numbers is to write a series of figures on the board, and to direct the pupil to add or multiply the first two pointed at, to subtract the next, to divide by the third, etc. Examples like these should present no difficulty:

$$(3 \times 29) - (4 \times 16) + 7;\ 10 \times 9 \times 3.$$

The teacher should always solve the examples mentally with the class.

Grube recommends also the following exercises at this part

of the course as a test whether the pupil has a clear or fixed idea of each number:

Let the pupils count from 1 to 100, but instead of naming the numbers themselves, name two factors of which each may be composed. Hence, instead of counting 6, 7, 8, 9, 10, the pupils are to say 2×3, 7×1, 2×4, 3×3, 2×5, etc. 1 is to be given as a factor only in case of prime numbers.

Numbers like 52, 68, 95, etc., must be remembered as the product of 4×13, 4×17, 5×19, etc.; and for this purpose the multiplication table up to 20 should be studied.

I. *c.* Somebody had $100 and spent the fourth part of it; of the remainder he spent the third part. What amount did he keep? What part of the $100 did he keep?

I have taken a number 3 times, and have 4 more than half a hundred. What number did I take three times?

Five times what number is 5 less than 100?

Seventy-five is three times the fourth part of what number?

Exercises in changing compound numbers within the limits of 100 to lower or higher denominations.

One quarter of a dollar has how many cents? Three quarters?

Half a dollar is how many quarters? Dimes?

A dollar is how many dimes?

How many months in 100 days? Weeks?

A hundred months are how many years?

One hundred pieces are how many dozen? Pairs?

One year and eight months are how many months?

One hundred ounces are how many pounds?

Eight pounds three ounces are how many ounces?

Twenty-three gallons are how many quarts?

One hundred quarts are how many gallons?

A farmer sold three mules for 99 dollars; how much apiece did he get for them? etc.

NUMBERS ABOVE ONE HUNDRED.

In teaching the numbers from 100 to 1,000, the transition is made to the ordinary four processes. Instruction gradually loses the character of an object lesson, and appeals to memory, understanding, and reason directly. Not that the help of illustrations is discarded altogether, for they should be used wherever feasible; but when dealing with larger numbers, the only way to illustrate is to show the analogy, with a corresponding example in smaller numbers, by which perception is enabled to help the higher powers of the mind. A few generalizations will be of assistance in following Grube's idea.

The number 100 is the last one treated by itself. With it, instruction proceeds no longer from one number to the next higher one, considering each number separately, but deals with the numbers from 100 to 1,000 in general.

Grube places the work with numbers from 100 to 1,000 in the first half of the third year of the course. The first quarter is devoted almost exclusively to pure number, the second more to applied number.

As the relation of the units and tens to each other has been considered in the previous course, the principal part of the work at this stage is the measuring of hundreds by hundreds, and of hundreds by tens.

The greater part of instruction here is oral work, or intellectual arithmetic; written work is but a repetition of the oral.

In the introduction to this division of his work, our author says, "As the future study of arithmetic is simply an application of the insight gained by perception into the nature of the numbers from 1 to 100, the following part of the course has for its purpose to reduce the relations of the numbers from 100–1,000 to those of the numbers below one hundred, or, in other words, to show that the relations of larger numbers among themselves are of the same nature as the relations of their elements."

By this practice the pupil arrives at the secret of excellence in performing examples mentally, — the dealing with numbers reduced to their smallest possible form.

In order to arrive at a true idea of number, we must look upon number itself at this stage, and not yet consider the four processes as such. The latter are reserved for the second half of the year. Intellectual and written arithmetic should always be combined.

As there is no longer any need for the isolated consideration of each number, as in the former part of the course, the only division of the subject-matter necessary is

A. The Pure Number (*measuring, comparing, combining*).

B. Applied Numbers.

Grube's six divisions of the work with pure number from 100 to 1,000 show the plan which he recommends; and after having given them, nothing of the peculiar features of his method remains except the teaching of fractions.

FIRST STEP.

NUMBERS FROM ONE HUNDRED TO ONE THOUSAND.

Measuring by the units of the Decimal system, by units, tens, and hundreds.

Illustrations should be used. Grube recommends solid blocks divided by lines into 10 and 100 units. Squares of paste-board will answer the same purpose.

EXERCISES: 768 = 7 hundreds, 6 tens, 8 units. The 8 units belong to the 7th ten of the 8th hundred; two units would complete the 7th ten, 3 tens more the 8th hundred, 2 hundreds more would complete 1,000.

Analyze in this way 500, 704, 174, 714, 829, 999, etc.

What number has 3 hundreds, 6 tens, 5 units?

How many units in 7 hundreds, 8 tens, 9 units?

How many units in 1,000? how many hundreds?

Written Work.

	Hundreds	Tens	Units
$615 = 6 \times 100 + 1 \times 10 + 5 \times 1 = 6$	1	5	
$204 = 2 \times 100 + 0 \times 10 + 4 \times 1 = 2$	0	4	
or $615 = 600 + 10 + 5$	etc.		

SECOND STEP.

HUNDREDS MEASURED BY HUNDREDS.

A. 200 (200; 100)

(Objective Illustration, — Measuring and Comparing, — Rapid Solution of Problems, — Combinations: The same as in the first part of the course.)

In the first part of the course the diagram under the number 2 was:

$$1 + 1 = 2$$
$$2 \times 1 = 2$$
$$2 - 1 = 1$$
$$2 \div 1 = 2$$

Hence the diagram of 200 measured by 100 is

$$100 + 100 = 200$$
$$2 \times 100 = 200$$
$$200 - 100 = 100$$
$$200 \div 100 = 2$$

What number is contained twice in 200? 100 is half of what number? What number must I double in order to have 200? etc.

B. *a.* 300 (300; 100) (300; 200)

(300; 100)
$$\begin{cases} 100 + 100 + 100 = 300 \\ 3 \times 100 = 300 \\ 300 - 100 - 100 = 100 \\ 300 \div 100 = 3 \end{cases}$$

(300; 200)
$$\begin{cases} 200 + 100 = 300 \\ 1 \times 200 + 100 = 300 \\ 300 - 200 = 100 \\ 300 \div 200 = 1\ (100) \end{cases}$$

300 is 100 more than 200, 200 more than 100.

200 is 100 less than 300, 100 more than 100.

100 is 200 less than 300, 100 less than 200.

300 is three times 100, 100 is the third part of 300.

b. $300 - 100 - 100 + 200 \div 100 = ?$ etc.

c. From what number can you take twice 100 and have a remainder of 100?

C. 400 (400 ; 100) (400 ; 200) (400 ; 300)

a. 1. MEASURING WITH 100.

$$\begin{cases} 100 + 100 + 100 + 100 = 400 \\ 4 \times 100 = 400 \\ 400 - 100 - 100 - 100 = 100 \\ 400 \div 100 = 4 \end{cases}$$

2. MEASURING WITH 200.

$$\begin{cases} 200 + 200 = 400 \\ 2 \times 200 = 400 \\ 400 - 200 = 200 \\ 400 \div 200 = 2 \end{cases}$$

3. MEASURING WITH 300.

$$\begin{cases} \left\{ \begin{array}{l} 300 + 100 = 400 \\ 100 + 300 = 400 \end{array} \right. \\ 1 \times 300 + 100 = 400 \\ 400 - 300 = 100 \end{cases}$$

4. MISCELLANEOUS MEASURING.

400 is 100 more than 300
200 more than 200
300 more than 100
300 is 100 less than 400
200 is 200 less than 400
100 is 300 less than 400

4 is contained in 4 once.
4 is contained in 400 a hundred times.
2 is contained in 4 twice.
2 is contained in 400 two hundred times.

b. 100 + 200 + 100 ÷ 200 = ? etc.

c. What number is twice 100 greater than 200? etc.

D. 500, (500 ; 100), (500 ; 200), (500 ; 300), (500 ; 400), etc.

E. 600, etc.

THIRD STEP.

MIXED HUNDREDS MEASURED BY MIXED HUNDREDS.

(This step is a variation of the preceding one. It is, of course, neither possible nor necessary to consider every number which consists of hundreds and tens, since all that is required here is a knowledge of how to perform the operation of comparing hundreds and tens with hundreds and tens. For this object a limited number of examples is sufficient.)

What number is 2, 3, 4, 5 × 110? 440 = 4 × 110, = 2 × 220, 660 = 6 × ? 3 × ? 880 = 8 × ? 4 × ? 2 × ? 990 = 9 × ? 3 × ? Of what factors may 888 be considered to consist? 999?

If 333 . . . divide 999 among themselves, how much will each part be? If 3 divide 999? If 2 divide 888?

Of what number is 120 the 3d part? the 4th? the 5th?

What number equals the fourth part of 844?

844 is four times what number? What number is contained 4 times in 844? Half of 844 is how many more than one-fourth of this number?

One-third of 333 is one-sixth of what number?

Compare 365 with 244. (365 = 3h + 6t + 5u; 244 = 2h + 4t + 4u; 3h − 2h = 1h; 6t − 4t = 2t; 5u − 4u = 1u; 365 − 244 = 1h + 2t + 1u; 365 is 121 more than 244; 244 is 121 less than 365.)

Difference between 743 and 120?

What number is equal to the sum of 743 + 221?

112 + 113 + 114 = ? 659 − 222 − 124 = ?

111 + 212 + 313 = ? etc.

FOURTH STEP.

MEASURING OF HUNDREDS BY TENS.

I. *a.* Pure hundreds.

If 100 = 10 × 10, then

2 × 100 or 200 = 2 × 10 × 10 = 20 × 10
3 × 100 or 300 = 3 × 10 × 10 = 30 × 10
4 × 100 or 400 = 4 × 10 × 10 = 40 × 10
10 × 100 or 1000 = 10 × 10 × 10 = 100 × 10

b. Hundreds and tens.

If 100 = 10 × 10,

110 = (10 × 10) + (1 × 10) = 11 × 10
120 = (10 × 10) + (2 × 10) = 12 × 10
130 = (10 × 10) + (3 × 10) = 13 × 10
990 = (90 × 10) + (9 × 10) = 99 × 10

c. Hundreds, tens, and units by tens.

If 100 = 10 × 10, then

101 = (10 × 10) + 1
109 = (10 × 10) + 9
906 = (90 × 10) + 6
814 = (81 × 10) + 4

How many tens in 500, 900? etc.
What number consists of 53 tens?
What number contains 9 units more than 53 tens?
How many times 10 in 660, 420, 870?
10 is the 42d, 66th part of what number?

II. Comparison of numbers.

Compare 400 with 900 as to the number of tens they contain. 55 tens are how many tens less than 600? 660? 990? 880 is composed of what 4 equal number of tens? 800 + 180 + 20 = ? 210 − 160 = ? 60 tens are how many hundreds? What number has 8 tens and 9 units more than 490? What number taken 87 times and 9 added to it is 879? How many tens more in 73 tens than in twice 240? The 100th part of 1,000 is contained how many times in 500? One-third of 630 is one-fourth of what number? The 68th part of 680 + the 24th part of 240 are how many less than 10 × 36?

The exercises are followed by examples which show that the factors in multiplication are interchangeable.

$$110 = 11 \times 10 = 10 \times 11$$
$$220 = 22 \times 10 = 10 \times 22$$
$$680 = 68 \times 10 = 10 \times 68$$

What number must I take 10 times in order to get 670? 67 times?

Of what number is 67 the tenth part? What is the 67th part of 670?

How many times is 79 contained in 790? What number can be taken ten times from 790? 79 times? 79 times ten is equal to 10 times what number?

FIFTH STEP.

MEASURING A NUMBER BY ITS FACTORS.

I. *a.* **Pure hundreds.**

$100 = 2 \times 50,\ 4 \times 25,\ 5 \times 20,$ etc.
$200 = 2 \times 2 \times 50 = 4 \times 50$
$200 = 2 \times 4 \times 25 = 8 \times 25$
etc., etc.

b. HUNDREDS AND TENS.

$220 = 10 \times 22$, and since $10 = 2 \times 5$,
$= 2 \times 5 \times 22 = 2 \times 110$, and since $22 = 2 \times 11$
$= 10 \times 2 \times 11 = 10 \times 22$, etc.

c. HUNDREDS, TENS, UNITS.

$$426 = (10 \times 42) + 6$$
$$= (4 \times 100) + 26, \text{ etc.}$$

II. What is the difference between 980 and 377?

The difference between 980 and 377 is three times what number?

By what number must I divide 365 to obtain five?

What difference between the 22d and the 30th part of 660?

SIXTH STEP.

REDUCTION OF NUMBERS FROM 1 TO 1,000 INTO THEIR ELEMENTS.

It is immaterial in what order the numbers are considered, or what numbers are taken up; the practice alone which these exercises afford to the pupil is important.

A pupil who has done the work of the previous course will be able to separate a number into its parts quickly and accurately. The teacher gives the number, and the pupils separate it orally or in writing.

360.

$300 + 60$	$(3 \times 100) + (3 \times 20)$
$180 + 180$	3×120
$200 + 160$	10×36
$320 + 40$	5×72
$336 + 24$, etc.	20×18, etc.

DIVISION OF THE WORK ACCORDING TO GRUBE.

3d year, 2d quarter: Compound numbers, money, weights, measures.

3d and 4th quarters, oral and written work: Numeration, Addition, Multiplication, Subtraction, and Division with any number according to the usual methods of analysis.

4th year, 1st term: Object lessons in fractions, on the same plan as the lessons with the numbers from 1 to 10 at the beginning of the course.

2d term. The four processes with fractions.

FRACTIONS.

Leaving the work with whole numbers, after having considered compound and applied numbers in the second quarter, and passing over the four species whose treatment is about the same as can be found in any other arithmetic, we shall find again an original and peculiar application of Grube's idea in the teaching of fractions.

The pupil is expected to take up this subject in the fourth year of the course after having acquired some knowledge of fractions by previous instruction.

"In the same way," says Grube, "in which the pupil arrived at the perception of whole numbers by measur-

ing them by the smallest unit, fractions are now explained to him by comparison with and reference to the number One, from which they have arisen.

"While the number one has appeared so far as a part of other numbers, it is now considered as a whole, which consists of parts. The latter in relation to this whole are called fractions."

As the pupils have already learned to look upon whole numbers as parts of larger numbers, the following method of teaching fractions will offer no special difficulty, since the process is the same as the one which has made them familiar with integers, and which consists in the perception of the manifold relations of the number which is being taught.

The order in which fractions are considered is, halves, thirds, fourths, fifths, etc. The processes to which fractions are subjected are again:

I. **Pure number,** and under this
   a. MEASURING.
   b. COMPARING.
   c. COMBINATIONS.

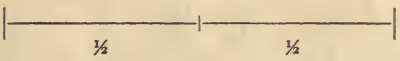
II. Application of what has been taught with pure numbers, in applied examples involving the four processes.

The regular illustration for fractions is the line divided into parts; a circle divided into parts may be substituted for it. It is necessary to give an abundance of practical examples under each fraction, since the four processes are explained and made use of at the very beginning. In Division with fractions, Grube

urges strongly not to go here beyond the idea of "being contained in." It is nonsense, he says, to speak of 2 divided by one-half, and the like, at this period of instruction. That $\frac{1}{2}$ is contained 4 times in 2 will be understood by the child, because it can be shown to him; but the idea of division is more difficult. Even examples like $4 \div \frac{2}{3}$ should not be read four divided by $\frac{2}{3}$, but rather, 4 is twice the third part of what number? or, still better, $\frac{2}{3}$ are contained in 4 how many times?

FIRST STEP.

HALVES.

1. 

If I divide one (a unit) into two equal parts, I obtain 2 halves. A half is one of the 2 equal parts into which I have divided the whole.

$$1 \div 2 = \tfrac{1}{2}, \text{ or } \tfrac{1}{2} \times 1 = \tfrac{1}{2}.$$

MEASURING.

a. (Addition.) $\frac{1}{2} + \frac{1}{2} = 1$.

b. (Multiplication.) $1 \times \frac{1}{2} = \frac{1}{2}$. $2 \times \frac{1}{2} = 1$.

c. (Subtraction.) $1 - \frac{1}{2} = \frac{1}{2}$.

d. (Division.) $\frac{1}{2} + \frac{1}{2} = 1$, $1 \div \frac{1}{2}$ ($\frac{1}{2}$ is contained 2 times in 1).

APPLICATIONS OF THESE FOUR EXAMPLES:

1. $1 \div 2 = \frac{1}{2}$, hence $2 \div 2 = \frac{2}{2}$, $3 \div 2 = \frac{3}{2}$,
$10 \div 2 = \frac{10}{2}$, $100 \div 2 = \frac{100}{2}$, etc.

a.

$\frac{1}{2} + \frac{1}{2} =$	$1\frac{1}{2} + \frac{1}{2} =$	$1\frac{1}{2} + 1\frac{1}{2} =$
$1 + \frac{1}{2} =$	$2\frac{1}{2} + \frac{1}{2} =$	$7\frac{1}{2} + 4\frac{1}{2} =$
$2 + \frac{1}{2} =$	$12\frac{1}{2} + \frac{1}{2} =$	$7\frac{1}{2} + 8 =$
$3 + \frac{1}{2} =$	$18\frac{1}{2} + \frac{1}{2} =$	$7\frac{1}{2} + 8\frac{1}{2} =$
etc.	etc.	etc.

b. $2 \times \frac{1}{2} = \frac{2}{2}$ $\quad 1 \times 1\frac{1}{2} = 1 \times \frac{3}{2} = 1\frac{1}{2}$

$3 \times \frac{1}{2} = \frac{3}{2} = 1\frac{1}{2}$ $\quad 2 \times 1\frac{1}{2} = 2 \times \frac{3}{2} = \frac{6}{2} = 3$

$10 \times \frac{1}{2} = \frac{10}{2} = 5$ $\quad 3 \times 1\frac{1}{2} = 3 \times \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$

$100 \times \frac{1}{2} = \frac{100}{2} = 50$ $\quad$ etc.

$6 \times 15\frac{1}{2} = (6 \times 15) + 6 \times \frac{1}{2}$, etc.)

$9 \times 80\frac{1}{2} =$

(If $\frac{1}{2} \times 1 = \frac{1}{2}$, then $\frac{1}{2} \times 6 = \frac{6}{2} = 3$, $\frac{1}{2} \times 9 = 4\frac{1}{2}$, etc.)

c.

$1 - \frac{1}{2} = \frac{1}{2}$	$2 - 1\frac{1}{2} = \frac{1}{2}$	$2\frac{1}{2} - 1 =$
$2 - \frac{1}{2} = 1\frac{1}{2}$	$6 - 4\frac{1}{2} = 1\frac{1}{2}$	$6\frac{1}{2} - 3 =$
$3 - \frac{1}{2} = 2\frac{1}{2}$	$9 - 3\frac{1}{2} =$	$3\frac{1}{2} - 2\frac{1}{2} =$
etc.	etc.	$8\frac{1}{2} - 4\frac{1}{2} =$

d. $1 \div \frac{1}{2} = 2$ (for $1 = \frac{2}{2}$, in $\frac{2}{2}$ one-half is contained twice, hence $1 \div \frac{1}{2} = 2$).

$4 \div \frac{1}{2} = 8$	$1\frac{1}{2} \div \frac{1}{2} = 3$	$6 \div 1\frac{1}{2} =$
$6 \div \frac{1}{2} =$ etc.	$9\frac{1}{2} \div \frac{1}{2} =$ etc.	

$10\frac{1}{2} \div 3\frac{1}{2} = \frac{21}{2} \div \frac{7}{2} = 21 \div 7 = 3.$

2. *a.* Compare $\frac{1}{2}$ with 1; $\frac{1}{2} = 1 - \frac{1}{2}$, $1 = \frac{1}{2} + \frac{1}{2}$, $\frac{1}{2}$ = half of 1, $1 = 2 \times \frac{1}{2}$.

b. What number is equal to the difference between $\frac{1}{2}$ and 1? How many must I take from 16 to obtain $9\frac{1}{2}$?

Of two numbers the smaller one is $9\frac{1}{2}$, the difference between it and the larger one is $6\frac{1}{2}$; what is the other number?

Name some other two numbers that have a difference equal to $6\frac{1}{2}$.

c. How many times must I take $\frac{1}{2}$ in order to have 1? $4\frac{1}{2}$ in order to have 9? 18? $4\frac{1}{2}$ is half of what number? 9 is twice what number?

The quotient is 2, the divisor $4\frac{1}{2}$; what is the dividend? (The quotient 2 tells that $4\frac{1}{2}$ must be contained 2 times in the divisor, hence the divisor must be twice $4\frac{1}{2} = 9$.) I must take one-half of what number in order to have $4\frac{1}{2}$? etc.

3. *a.* What is meant by $\frac{1}{2}$ dollar? dozen? (One-half dollar is one of the two equal parts into which a dollar may be divided.)

b. How many half dollars in 55 cents? $\frac{1}{2}$ dollar + 5 cents, etc.

c. Difference between 8 times 55 cents and 9 times 57 cents?

$$(8 \times 55c = 8 \times \$\tfrac{1}{2} + 8 \times 5c = \$4, 40c.$$

$$9 \times 57c = 9 \times \$\tfrac{1}{2} + 9 \times 7c = \$4\tfrac{1}{2} + 63c$$
$$= \$4\tfrac{1}{2} + \$\tfrac{1}{2} + 13c = \$5.13.$$

$\$5.13 - \$4.40 = 73c$, hence the difference, etc.)

d. The cook of a hotel buys $17\frac{1}{2}$ pounds of meat + $13\frac{1}{2}$ pounds + $8\frac{1}{2}$ pounds. This will be sufficient for how many persons if 8 ounces are the calculated allowance for each?

e. If a pound of tea costs $\frac{1}{2}$ dollar, how much can be bought for 25 cents?

f. If 5 yards of cloth cost 6 dollars, what is the price of $10\frac{1}{2}$ yards? (1 yard = fifth part of \$6 = \$1 + fifth part of 100 cts. = \$1.20. $\frac{1}{2}$ yard = 60c. 10 yards = \$12. $10\frac{1}{2}$ yards = \$12.60.)

3. Applied examples.

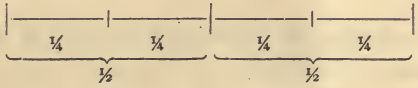
In the treatment of the other fractions, the same plan is followed. Fourths, for instance, are first compared with the whole, then with halves, by addition, multiplication, subtraction, division, and finally with thirds. In the latter process, the illustration is peculiar, and consists of two parallel horizontal lines drawn close to each other, the upper one divided into four parts, the lower one into three parts, and then each line by light marks again into twelve parts, so that both show the mediating fraction of twelfths and their relation to fourths and thirds.

The following is a brief abstract of the treatment of fourths, giving in full those details only which cannot be understood from what has been said in connection with the treatment of $\frac{1}{2}$.

THIRD STEP.

FOURTHS.

A. Fourths, Halves, and Units.



1. If I divide 1 into 4 equal parts, each part, etc.

$$1 \div 4 = \tfrac{1}{4}, \text{ or } \tfrac{1}{4} \times 1 = \tfrac{1}{4}.$$

a. $\frac{1}{4} + \frac{1}{4} = ?$ $\frac{2}{4} + \frac{1}{4} = ?$ etc. (Adding by fourths.)

b. $1 \times \frac{1}{4} = ?$ $2 \times \frac{1}{4} = ?$ etc. (Multiplying by fourths.)

c. $1 - \frac{1}{4} = ?$ $\frac{3}{4} - \frac{1}{4} = ?$ etc. (Subtracting by fourths.)

d. $\frac{1}{4} \div \frac{1}{4} = ?$ $\frac{1}{2} \div \frac{1}{4} = ?$ etc. (Dividing by fourths.)

e. 1. Fourths as the quotient of integers: $1 \div 4 = \frac{1}{4}$, $2 \div 4$, etc.

2. As the product of fourths and integers: $\frac{1}{4} \times 3$, $\frac{1}{4} \times 100 = \frac{100}{4}$.

a. Addition (1. Mixed numbers and fourths, $4\frac{1}{4} + \frac{3}{4}$; 2. Mixed numbers + mixed numbers, $4\frac{1}{2} + 4\frac{1}{4}$).

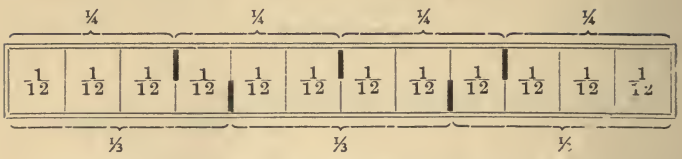
b. Multiplication (integers × fourths and × mixed numbers, etc.).

c. Subtraction.

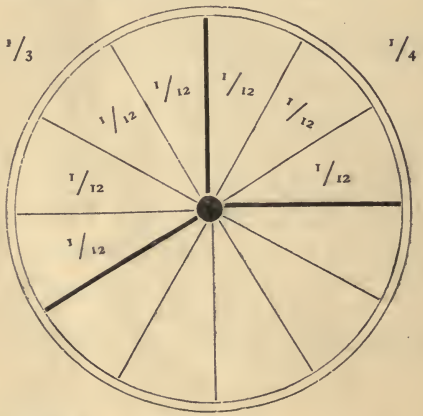
d. Division.

B. Fourths and Thirds.

ILLUSTRATION:



Or, if preferred, the circle may be used to illustrate the same principle, as follows:



1. Fourths and thirds meet in twelfths.

$\frac{1}{4} = \frac{3}{12}$, $\frac{1}{3} = \frac{4}{12}$.
$\frac{1}{4} = \frac{1}{3} - \frac{1}{12}$, $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$.
$\frac{1}{4} = \frac{3}{4} \times \frac{1}{3}$, for $\frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$, $(\frac{3}{12} \div \frac{4}{12} = 3 \div 4)$.
$\frac{1}{3} = \frac{4}{3} \times \frac{1}{4}$, for $\frac{1}{3} \div \frac{1}{4}$, etc.

2. Compare $\frac{1}{4}$ with $\frac{2}{3}$. $\frac{1}{4} = \frac{3}{12}$, $\frac{2}{3} = \frac{8}{12}$.
$\frac{1}{4} = \frac{2}{3} - \frac{5}{12}$, $\frac{2}{3} = \frac{1}{4} + \frac{5}{12}$.
$\frac{1}{4} = \frac{3}{8} \times \frac{2}{3}$ (the 8th part of $\frac{2}{3}$ taken 3 times; see illustration), for $\frac{1}{4} \div \frac{2}{3} = \frac{3}{8}$ (the 8th part of $\frac{2}{3}$ $(= \frac{}{8})$ is contained 3 times $(\frac{3}{})$ in $\frac{1}{4} = \frac{3}{12} \div \frac{7}{12} = 3 \div 8$).

$\frac{2}{3} = \frac{8}{3} \times \frac{1}{4}$, for $\frac{2}{3} \div \frac{1}{4} = \frac{8}{3}$. (The third part of one-fourth ($\frac{1}{12}$) is contained 8 times in $\frac{2}{3}$.)

3. Compare $\frac{3}{4}$ with $\frac{2}{3}$, etc.
4. Compare halves, fourths, and thirds.
5. Fractions, integers, and mixed numbers.
6. Combinations and rapid solution of problems.

C. *a.* **Applied numbers with fourths.**
b. Applied numbers with halves, thirds, and fourths.
c. Examples in analysis.
d. Miscellaneous examples.
The other fractions are treated in a similar way.

In giving an outline of Grube's method of teaching the elements of arithmetic, no attempt has been made to comment on any part of it, as it seemed desirable to submit the whole system as originally set forth to the judgment of practical teachers. Many points are open to criticism, and not a few may be obvious mistakes. A great number of text-books in arithmetic have been written in the country in which Grube's work was first published, which have improved the original method, and adapted it to the special wants of different school systems. It seemed better, however, to present the method as it was originally conceived, without giving expression to criticism and difference of opinion, and to let the well-known skill and ingenuity of the teachers of our common schools adapt it to our peculiar wants, and make such improvements and changes as may seem expedient.

In regard to one point of the system, however, it looks as if there could be no mistake. The thoroughness with which illustrations are used is an indispen-

sable condition for successful work in the primary grades. If the introduction of the kindergarten has taught some lessons to all of us, the least important among them is certainly not the remarkable results accomplished in arithmetic, when it is taught incidentally, by means of the building-blocks of Frœbel's "gifts." The writer has visited a kindergarten in which problems like "how many twenty-sevenths in three-ninths?" were solved by children five or six years old without any perceptible difficulty. The explanation of this proficiency lies certainly in the fact that ninths and twenty-sevenths are, for those children, not abstract terms, but names of some of the little cubes in their toy-box, and that ninths and twenty-sevenths are the names by which they know those little objects with whose comparative size long use has made them perfectly familiar. The association of arithmetical ideas with perceptible objects alone makes arithmetic intelligible to the child.

There can be no doubt that many of the methods of instruction used in the kindergarten are excellent and very suggestive, and should be carried over the primary grades as far as the character of the schoolroom, which must be kept distinct from that of a kindergarten, admits. In the common school, children learn by the senses of hearing and seeing; in the kindergarten by seeing, hearing, and touch. The hand is a very important means of education; and it seems evident that pupils in the primary grades, who are allowed to handle suitable objects, in arithmetic, to count them, to arrange them so as to represent the problems given to the school, will be able to do better work than if instruction in this important study is imparted without the help of objective illustrations.

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